A method for predicting the stability characteristics of three-term homogeneous recurrence relations

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ABSTRACT

A simple method is presented which determines the numerical stability or instability in the computation of any solution of a three-term homogeneous linear difference equation of order m in terms of the size of a single parameter. The method is illustrated by application to a second order and to a third order difference equation. Excellent agreement with predictions is reported.

1. INTRODUCTION

In this work we are concerned with predicting the behaviour of the fundamental solutions of the three-term m'th order homogeneous recurrence relation, written in the form

$$a_n(x)\,f_{n+m-i}(x) = b_n(x)\,f_n(x) + c_n(x)f_{n-i}(x), \quad (1.1)$$

in which $m \geq 2$, $i \geq 1$ and $a_n b_n c_n \neq 0$.

Recurrence relations have been extensively studied in recent years [1, 2] including three-term relations [3, 4].

Our method depends on making a balance between different pairs of terms in (1.1) and neglecting the remaining term. By assuming that the coefficients remain *locally constant* we find a single parameter ϵ in terms of which the stability characteristics of the solutions may be determined. For $\epsilon \ll 1$ the m solutions to (1.1) divide into two groups with m-i and i members respectively. The first group dominates the second in forward recursion whilst the reverse is true in backward recursion. Within each group interaction is weak and no specific solution dominates the others. Thus any solution within a dominant group can be computed by suitable choice of starting values. For $\epsilon \gg 1$ all m solutions are effectively uncoupled and stability is ensured in this regime for all solutions in either recursive direction. The method is illustrated by reference to a second order and to a third order recurrence relation.

2. METHOD

The basic idea is to simplify (1.1) by neglecting one term and solving the two-term relation that remains. This may be done in three ways. From each balance equation we can determine the relative behaviour of a group of solutions together with the range of ϵ over which this behaviour is valid. The balance equations are

$$f_{n+m-i}/f_n \approx b_n/a_n \quad \text{iff} \quad |f_{n-i}/f_n| \ll |b_n/c_n|, \quad (2.1)$$

$$f_{n-i}/f_n \approx -b_n/c_n \quad \text{iff} \quad |f_{n+m-i}/f_n| \ll |b_n/a_n|, \quad (2.2)$$

and

$$f_{n+m-i}/f_{n-i} \approx c_n/a_n \quad \text{iff} \quad |f_n/f_{n-i}| \ll |c_n/b_n|. \quad (2.3)$$

Now we assume that the coefficients a_n, b_n, c_n may be considered to be locally constant, which implies that the fundamental solutions may be approximated by the form

$$f_n \approx c\,\lambda^n, \quad (2.4)$$

for some λ, with c constant. This enables the growth of the set (2.1) to be expressed as

$$|\lambda| = |f_{n+1}/f_n| \approx |b_n/a_n|^{1/(m-i)}, \quad (2.5)$$

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whenever

$$\lambda^{-i} \ll |b_n/c_n|. \quad (2.6)$$

This inequality, with λ substituted from (2.5), may be alternatively written $\epsilon^{1/(m-i)} \ll 1$, where

$$\epsilon = \left| \frac{a_n}{b_n} \right|^i \left| \frac{c_n}{b_n} \right|^{m-i} \quad (2.7)$$

By similar reasoning the set (2.2) behaves like

$$|\lambda| = |f_{n+1}/f_n| \approx |c_n/b_n|^{1/i}, \quad (2.8)$$

whenever $\epsilon^{1/i} \ll 1$. We distinguish between solutions of type (2.5) and (2.8) by writing

$$u_{j,n+m-i}/u_{j,n} \approx b_n/a_n, \; j = 1, 2, \ldots, m-i, \quad (2.9a)$$

$$v_{k,n}/v_{k,n-i} \approx -c_n/b_n, \quad k = 1, 2, \ldots, i, \quad (2.9b)$$

both groups (totalling m solutions) being valid for $\epsilon \ll 1$. However, solutions to (2.3) form a single group denoted by

$$w_{k,n+m-i}/w_{k,n-i} \approx c_n/a_n, \quad k = 1, 2, \ldots, m, \quad (2.10)$$

and valid when

$$|\lambda|^i \ll |c_n/b_n|, \; |\lambda| \approx |c_n/a_n|^{1/m}. \quad (2.11)$$

The inequality in (2.11) is equivalent to $\epsilon^{-1/m} \ll 1$, which implies that the solution group (2.10) is valid in the regime $\epsilon \gg 1$.
We note also from (2.5), (2.8) and (2.9) that

$$\left| \frac{v_{k,n+1}}{v_{k,n}} \Big/ \frac{u_{j,n+1}}{u_{j,n}} \right| = O(\epsilon^{1/i(m-i)}), \quad (2.12)$$

a relation that expresses the relative behaviour of the $v_{k,n}$ to the $u_{j,n}$ growth rates. Thus in the regime $\epsilon \ll 1$, (2.12) shows that the $u_{j,n}$ group of solutions dominates the $v_{k,n}$ group in *forward recursion,* whilst the reverse is true for backward recursion [5]. Within the same group, interaction is very weak, and we can select any particular solution by suitable choice of the starting values. By contrast, within the regime $\epsilon \gg 1$, each member of the single group of solutions $w_{k,n}$ (2.10) is stable in both forward and backward recursion. Where ϵ remains 0(1) all three terms in (1.1) are of equal importance. This regime ($\epsilon \sim 1$) lasts only for a limited n (except in the special case in which ϵ is independent of n), for which it is sufficient to take a limited number of guarding figures.

3. APPLICATION TO THE BESSEL FUNCTIONS

This second order recurrence may be written

$$f_{n+1}(x) = (2n/x)\, f_n(x) - f_{n-1}(x), \quad (3.1)$$

from which, in comparison with (1.1), we set $i = 1$, $m = 2$, $a_n = 1$, $b_n = 2n/x$ and $c_n = -1$. The parameter ϵ (2.7) becomes

$$\epsilon = (x/2n)^2. \quad (3.2)$$

According to (2.9) the solutions behave like

$$u_{1,n+1}/u_{1,n} \approx 2n/x, \qquad \epsilon \ll 1, \quad (3.3a)$$

$$v_{1,n+1}/v_{1,n} \approx \frac{1}{2}x/(n+1), \; \epsilon \ll 1, \quad (3.3b)$$

whilst from (2.10)

$$w_{k,n+1}/w_{k,n-1} \approx -1, \; k = 1, 2, \; \epsilon \gg 1. \quad (3.4)$$

Solutions $w_{1,n}$ and $w_{2,n}$ are effectively uncoupled from each other.
Comparison of (3.3) with the known solution $J_n(x)$ and $Y_n(x)$ of the Bessel equation shows that $u_{1,n} = Y_n(x)$ and $v_{1,n} = J_n(x)$. Thus, for $\epsilon \ll 1$, Y_n ought to be computed by *forward recursion* and J_n by *backward recursion.* Indeed whatever starting conditions are given, we would expect to generate a multiple of Y_n in forward recursion, and a multiple of J_n in backward recursion. For $\epsilon \gg 1$ however recursion is stable in both directions for both J_n and Y_n.
The predictions were tested by numerical calculation. Relation (3.1) was used both in forward and backward recurrence to evaluate $J_n(x)$ and $Y_n(x)$, $n = 1, 2, \ldots, 20$, for two different values of x, namely 1 and 100.
From (3.2)

$$\epsilon(x=1) = 1/4n^2 \ll 1; \;\; \epsilon(x = 100) = (50/n)^2 \gg 1, \qquad n = 1, \ldots, 20. \quad (3.5)$$

yielding regimes with small and large ϵ respectively. The predictions of the theory were confirmed in every respect by our computations. We do not reproduce the known results here because Bessel functions have been studied and tabulated elsewhere [3, 5]. We note from (3.3) that the growth rates are not precisely independent of n but vary slowly with n for moderate values onwards. Parameter ϵ also varies with n and in this case, for any fixed x and large enough n, will take values small compared to unity.

4. APPLICATION TO A THIRD ORDER RECURRENCE

Three-term third order recurrence relations occur less frequently than do second order relations. Into this former category come, for example, repeated integrals of error functions and Bessel functions [6]. We consider a different example, that has important applications in rarefied gas dynamics,

$$2f_{n+2}(x) = (n+1)f_n(x) + x\, f_{n-1}(x). \quad (4.1)$$

Of the three independent solutions one must be always positive and represent the integral function [6, 7],

$$T_n(x) = \int_0^\infty t^n \exp(-t^2 - x/t)dt, \quad x > 0. \tag{4.2}$$

Comparing (4.1) with (1.1) we set $m = 3$, $i = 1$, $a_n = 2$, $b_n = n+1$ and $c_n = x$, from which (2.7) yields

$$\epsilon = 2x^2/(n+1)^3. \tag{4.3}$$

Where $\epsilon \ll 1$, two uncoupled solutions are given by (2.9a)

$$u_{j,n+1}/u_{j,n-1} \approx \frac{1}{2}(n+1), \qquad j = 1, 2, \tag{4.4a}$$

and the third solution from (2.9b)

$$v_{1,n}/v_{1,n-1} \approx -x/(n+1). \tag{4.4b}$$

In particular, for $n > 0$, relation (4.4a) yields

$$u_{1,n}/u_{1,n-1} \approx (\frac{1}{2}n)^{1/2}, \quad u_{2,n}/u_{2,n-1} \approx -(\frac{1}{2}n)^{1/2}, \tag{4.5}$$

from which

$$u_{2,n} \approx (-1)^n c\, u_{1,n}, \tag{4.6}$$

c being an arbitrary constant. When $\epsilon \gg 1$, the $w_{j,n}$ solution group (2.10) uncouples and

$$w_{j,n+2}/w_{j,n-1} \approx \frac{1}{2}x, \qquad j = 1, 2, 3, \tag{4.7}$$

from which we infer their common behaviour

$$w_{j,n+1}/w_{j,n} \approx (\frac{1}{2}x)^{1/3}. \tag{4.8}$$

It was shown [7] that $T_n(x)$ coincides with the solution $u_{1,n}$ for $n > 0$. Thus in the regime $\epsilon \ll 1$, since $u_{j,n}$ dominates $v_{1,n}$ for n increasing, we must use forward recursion to compute $T_n(x)$. The solution $u_{2,n}$ may also be computed by forward recursion if the correct starting values are supplied, there being no significant coupling between $u_{1,n}$ and $u_{2,n}$. When $\epsilon \gg 1$, the three $w_{j,n}$ solutions being uncoupled (4.7), each solution is stable in either recursive direction. For numerical testing purposes, two x-values were again chosen to ensure that ϵ was either small or large compared with unity for a given range of n. From (4.3)

$$\begin{aligned} &\epsilon(x = 0.1) = 0.02/(n+1)^3 \ll 1; \\ &\epsilon(x = 30) = 1800/(n+1)^3 \gg 1, \quad n = 1, 2, \ldots, 10. \end{aligned} \tag{4.9}$$

As in the example of Bessel functions, one of the coefficients of the recurrence (4.1) does change slowly with n. Nevertheless the essential features described in section 2 are upheld, and all predictions were confirmed by our computations. The function $T_n(x)$ has already been tabulated in [7] and we do not repeat the values here.

5. CONCLUSION

We have presented a simple method which proves effective in predicting the behaviour of the solutions to a three-term homogeneous recurrence relation. The method determines the stability or instability (dominance or otherwise) of all m solutions in forward and backward recursion in terms of a single parameter, whose magnitude, as n and x vary, provides the key to the explanation of the process.
It appears from our analysis that not only the relative rate of growth of the various solutions is significant, but also their relative coupling, that is whether one of the three terms in the recurrence might or might not be dropped.
Our method also predicts when it is permissible to perform computation with the use of starting guesses and when it is not. If, for example, there are two dominating solutions forwards and one dominating solution backwards, as in §4, we could use guessed starting values *only* for the last solution. The accurate values can then be recovered by use of a normalizing condition.

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